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Solving the n-Player Tullock Contest

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Abstract. The n-player Tullock contest with complete information is known to admit explicit solutions in special cases, such as (i) homogeneous valuations, (ii) constant returns, and (iii) two contestants. But can that model be solved more generally? In this paper, we show that key characteristics of the equilibrium, such as individual efforts, winning probabilities, and payoffs cannot, in general, be expressed in terms of the primitives of the model using basic arithmetic operations plus the extraction of roots alone. In this sense, the Tullock contest is intractable. We argue that our formal concept of tractability captures the intuitive understanding of the notion.

Keywords. Tullock contest · Pure-strategy Nash equilibrium · Solution by radicals · Galois theory

JEL codes. C02, C72, D72

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1 Preliminaries

1.1 Introduction

In the standard n-player Tullock (1980) contest with complete information, each contestant $i \in \{1, \ldots, n\}$ independently chooses an effort level $x_i \geq 0$ so as to maximize the payoff function

$$\Pi_i = \frac{x_i^R}{x_1^R + \ldots + x_n^R} V_i - x_i, \tag{1}$$

where $V_i > 0$ is contestant i's valuation of winning, and $R > 0$ is a parameter of the model that measures the extent to which effort, rather than luck, determines the winner of the contest.[1] As usual, the ratio in equation (1) is understood to assume the value $\frac{1}{n}$ if the denominator vanishes. Moreover, by renaming the contestants if necessary, it may be assumed w.l.o.g. that $V_1 \geq \ldots \geq V_n$. The game described above has found widespread applications in various areas such as marketing, lobbying, electoral competition, and sports (Konrad, 2009).[2]

One of the reasons why Tullock's model has been so fruitful is that its equilibrium in pure strategies admits a convenient representation in important special cases. Solution formulas are available if (i) valuations are homogeneous, i.e., $V_1 = \ldots = V_n$, (ii) returns from effort are constant (i.e., $R = 1$), and (iii) there are $n = 2$ contestants. In those cases, the system of necessary Karush-Kuhn-Tucker conditions for an optimal choice of effort,

$$\frac{\partial \Pi_i}{\partial x_i} = \frac{R x_i^{R-1}(x_1^R + \ldots + x_{i-1}^R + x_{i+1}^R + \ldots + x_n^R)}{(x_1^R + \ldots + x_n^R)^2} V_i - 1 \leq 0,$$

[1] Indeed, in the limit case $R \searrow 0$, each contestant wins with the same probability $\frac{1}{n}$ regardless of efforts, whereas in the other limit case $R \nearrow \infty$, the highest effort wins with certainty, just as in the all-pay auction.

[2] In line with the relevant parts of the literature, we will assume that contestants differ in valuations only. All our results can, however, be readily adapted to a setup where contestants may differ, in addition, in abilities (i.e., individual weight factors put in front of the x_i^R terms) and marginal costs.

with equality if $x_i > 0$, turns out to be tractable under suitable restrictions on the remaining parameters. However, it has remained an open question for some time if further generalization is feasible.[3]

This paper provides evidence showing that the search for further generalization is, in a sense, bound to be futile. To this end, we show that the n-player Tullock contest with heterogeneous valuations and non-constant returns cannot, in general, be *solved by radicals*. That is, it is not feasible to express endogenous characteristics of the equilibrium, such as efforts, winning probabilities, or payoffs, in terms of the primitives of the model using basic arithmetic operations such as addition, subtraction, multiplication, and division, as well as the extraction of roots alone. We also argue, but cannot prove of course, that our formal definition of tractability is equivalent to what is intuitively understood by the notion.[4]

The derivation of our main result, the intractability of the Tullock contest, proceeds in two steps. The first step is simple. Recalling that the probabilistic contest is an aggregative game,[5] we combine the necessary first-order conditions to a single polynomial equation whose unique positive solution pins down the equilibrium values of individual efforts, winning probabilities, and payoffs for all players. The issue of tractability of the Tullock contest is thereby boiled down to the question if a polynomial equation can be solved by radicals or not.[6] The second step of the analysis, however, is based on

[3]Cf. the discussion in Ryvkin (2007). The lack of a complete solution complicates, in particular, the comparison with the standard all-pay auction for which a complete solution is available (Baye et al., 1996).

[4]The term "radical" appears also in Hilbert's Nullstellensatz (Kubler et al., 2014, Thm. 2.1). There, it refers to the radical of an ideal, whereas here, it corresponds to the extraction of an N-th root.

[5]In an *aggregative* game, individual payoffs are functions of the player's own action and some aggregate of the actions of the other players. See, e.g., Corchón (1994).

[6]Thus, the initial step of our analysis is analogous to the application of the *Shape Lemma* (Kubler and Schmedders, 2010a) that finds, under general conditions, a single univariate polynomial equation from which the solutions to a whole system of polynomial equations in several variables can be derived

tools from *Galois theory*, which is a core topic in abstract algebra (van der Waerden, 1931). Specifically, we construct an example with $n = 5$ contestants and parameter $R = \frac{1}{2}$, and show that the associated polynomial equation derived in the first step cannot be solved by radicals. That result is extended, first to general valuations, and then to an arbitary number of contestants $n \geq 5$. As any general solution formula for n contestants must solve, in particular, the case $R = \frac{1}{2}$, we obtain our impossibility result.[7]

1.2 Galois theory

In the early 19th century, the French mathematician Évariste Galois developed the theory named after him, concomitantly with the theory of permutation groups, to address questions of tractability of polynomial equations of degree five and beyond.[8] The idea of his theory is that the roots of a polynomial constitute a finite set whose elements can be permuted, i.e., reshuffled like a deck of cards.[9] The *Galois group* of a polynomial consists of those permutations of the roots that leave all multivariate algebraic relationships with rational coefficients between the roots intact. The fundamental insight of Galois was that the structure of the Galois group of a polynomial $g(X)$ contains information about the solvability of the associated polynomial equation $g(X) = 0$ by radicals. That is, the Galois group encodes if, and if so how, the roots of a polynomial equation can be computed from the coefficients of the polynomial using the basic

in a straightforward way.

[7] While this settles the issue in the general case, we will also explain why it is unlikely that solution formulas for other values of R will be found.

[8] In fact, at the time, Galois' theory allowed solving other long-standing problems related to the trisection of the angle, the doubling of the cube, and the construction of regular polygons (Edwards, 1984). More recently, Galois representations featured prominently in Andrew Wiley's proof of Fermat's last theorem.

[9] As usual, a complex number $z \in \mathbb{C}$ is called a *root* of the polynomial $g(X)$ if $g(z) = 0$.

arithmetic operations of addition, subtraction, multiplication, and division, plus the extraction of roots. To obtain a definite answer for a specific polynomial equation, it is crucial to check:

(i) if the polynomial is *irreducible*, i.e., it does not decompose into a product of simpler polynomials, and

(ii) provided that the polynomial is irreducible, that the Galois group of the polynomial is *solvable*.[10]

Thus, if the relevant polynomial is irreducible and the Galois group lacks the property of solvability, one may conclude that its roots cannot be represented "in explicit terms." In this paper, we use Galois theory to study the tractability of the n-player Tullock contest.

1.3 Contribution

To evaluate the contribution of the present paper, it is essential to understand why our main result, the intractability of the Tullock contest, is not a straightforward consequence of the *Abel-Ruffini impossibility theorem.*[11] In short, that theorem says that, in contrast to polynomials of degree at most four, there is no general solution to polynomial equations of degree five or higher. On a superficial level, that seems to settle our research question, because it is not too difficult to find specifications of the primitives of the n-player Tullock contest for which the equilibrium conditions can be combined into a polynomial equation of arbitrarily high degree. However, there do

[10] Definitions will be provided below.

[11] Cf. Kubler et al. (2014, fn. 5). An accessible exposition of the Abel-Ruffini theorem can be found in Rosen (1995). Dummit (1991) and Kobayashi and Nakagawa (1992) derived formulas for the roots of solvable equations of degree five. Spearman and Williams (1994) characterized such equations.

exist families of polynomial equations of degree five and above that admit an explicit solution. And in fact, this matters in the case of the Tullock contest, as the following examples illustrates.

Example 1 *Suppose that for $n = 5$ contestants, valuations are given by*

$$(V_1, V_2, V_3, V_4, V_5) = (27, 18, 12, 7, 2).$$

Suppose also that $R = \frac{1}{2}$. Then, as detailed in Appendix A.1, the analysis of the first-order conditions leads to a quintic, i.e., to a polynomial equation of degree five. Still, the unique equilibrium is given by $x_i^* = \frac{24}{(1+48/V_i)^2}$, or more explicitly, by

$$\begin{aligned}(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) &= (\tfrac{1944}{625}, \tfrac{216}{121}, \tfrac{24}{25}, \tfrac{1176}{3025}, \tfrac{24}{625}) \\ &= (3.11, 1.79, 0.96, 0.39, 0.04).\end{aligned}$$

The example shows that there exist tractable examples even in cases in which the analysis leads to a polynomial equation of degree larger than four. Thus, the main result of the present paper indeed does not easily follow from the Abel-Ruffini theorem.

1.4 Related literature

The present paper lies on the intersection of two strands of literature. The first is the literature on contests. Since Tullock's (1980) seminal work, explicit solutions of the probabilistic contest model have been derived under various sets of assumptions. Key contributions identifying pure-strategy solutions in the basic model, such as Hillman and Riley (1989), Pérez-Castrillo and Verdier (1992), Nti (1999), Stein (2002), and Cornes and Hartley (2005), will be reviewed in the next section.[12] Most prominently,

[12] Pérez-Castrillo and Verdier (1992) and Cornes and Hartley (2005) documented the possibility of multiple pure-strategy equilibria in Tullock contests with increasing returns and more than two players. Mixed-strategy equilibria of the Tullock contest have been characterized by Baye et al. (1994), Alcalde and Dahm (2010), Ewerhart (2015, 2017a, 2017b), and Feng and Lu (2017).

Ryvkin (2007) pointed out the unavailability of explicit solutions for the n-player Tullock contest with heterogeneous valuations and non-constant returns. To address the issue, he employs first-order Taylor approximations around the tractable case of homogeneous valuations.[13]

The second strand of literature concerns the use of algebraic methods in game theory. Nash and Shapley (1950) characterized equilibria in behavior strategies in terms of roots of polynomial equations. In seminal work, Blume and Zame (1994) pointed out that the set of sequential equilibria of a finite extensive-form game is semi-algebraic, i.e., it may be understood as the set of solutions of a system of polynomial equalities and inequalities. Nau et al. (2004) noted that an irrational Nash equilibrium of a finite normal-form game with rational coefficients cannot be a corner point of the set of correlated equilibria. Datta (2003) showed that any real algebraic variety may be understood as the set of totally mixed equilibria of some finite normal-form game. Kubler and Schmedders (2010a, 2010b) proposed constructing Gröbner bases for semi-algebraic sets that characterize equilibria of various kinds. Nie and Tang (2024,2024) obtained algorithmic solutions to Nash equilibrium problems that are given by polynomial functions. None of those papers, however, employed Galois theory. As far as we know, there have been very few applications of Galois theory to game theory so far. Specifically, in response to the question by McKelvey and McLennan (1997) as to whether the computation of the equilibrium set could be simplified by starting from one equilibrium considered as known, Gandhi and Chatterji (2015) used Galois theory to construct new equilibria from a given sample equilibrium.[14] That approach led, in particular, to novel algorithms for the computation of mixed Nash equilibria in games

[13]See also Ryvkin (2013).

[14]See also Gandhi (2011) and Chatterji and Gandhi (2011).

with rational payoffs and irrational equilibria. However, those contributions are not directly related to our impossibility result.[15]

1.5 Overview

The remainder of the paper is structured as follows. Section 2 provides the necessary background on Galois theory. Section 3 reviews existing equilibrium characterizations for the Tullock contest. Section 4 presents our main result. Section 5 offers some discussion. Section 6 concludes. An Appendix contains material omitted from the body of the paper.

2 Background on Galois theory

This section provides the necessary background on Galois theory. We will discuss polynomial equations (Subsection 2.1), the Galois group (Subsection 2.2), and the Galois equivalence (Subsection 2.3).

2.1 Polynomial equations

We will consider equations of the type $g(X) = 0$, where

$$g(X) = a_N X^N + \ldots + a_1 X + a_0$$

is a *polynomial* with rational coefficients $a_0, \ldots, a_N$.[16] If $a_N \neq 0$, then N is the *degree* of $g(X)$.

[15] The notion of algebraic tractability developed below also clearly differs from the widely used notion of computational tractability, which refers to deterministic computability in polynomial time in the context of the P vs. NP-problem (e.g., von Stengel and Forges, 2008).

[16] Thus, we restrict attention to polynomials over the field of rational numbers $\mathbb{Q} = \{\frac{p}{q} : p, q \text{ integers}, q \neq 0\}$.

A polynomial equation $g(X) = 0$ is said to be *solvable by radicals* if its roots can be found from the coefficients by repeatedly taking sums, differences, products, and quotients, as well as by extracting roots. Here, the individual operations are understood to be finite in nature so that, e.g., infinite series are not allowed. Further, one should note that the definition does not change if we allow the formula to make use of arbitrary rational numbers as well, because those than be computed easily from any nonzero coefficient using the beforementioned arithmetic operations. Finally, the extraction of roots refers to the inverse of the power map $z \mapsto z^N$, where $N \geq 2$ is an integer.[17]

For instance, the polynomial equation $X^5 - 2 = 0$ is solvable by radicals because one solution is given by $X = \sqrt[5]{2}$, and the other solutions can be easily found by multiplying the known solution with a fifth unit root. In contrast, $X^5 - X + 1 = 0$ is a classic example of a quintic not solvable by radicals. Proving that a specific equation is not solvable, however, requires the methods of Galois theory that will be reviewed below.

2.2 The Galois group

To decide if a polynomial equation is solvable, it is obviously sufficient to restrict attention to polynomials that cannot be easily factored. Formally, a polynomial is called *irreducible* if it cannot be written as a product of two or more polynomials of degree at least one with rational coefficients.[18] A useful and well-known fact is

[17]For positive $z > 0$, the inverse map is simply $z \mapsto \sqrt[N]{z}$. For complex z, however, the power map is not globally invertible. Instead, any $z \neq 0$ admits precisely N pre-images that differ by powers of the N-th unit root. This multiplicity issue must be kept in mind when extracting roots from complex numbers.

[18]While the irreducibility of a given polynomial may, with some luck, be verified by hand using Gauss' lemma, parameter transformations, and Eisenstein's criterion, we used a computer algebra

that the roots of a polynomial $g(X)$ that is irreducible over $\mathbb{Q}$ have multiplicity one (Stewart, 2015, Prop. 9.14). Thus, the number of distinct roots equals the degree of the polynomial.

Consider an irreducible polynomial $g(X)$ of degree N with rational coefficients. Let $r_1, \ldots, r_N$ denote the different roots of $g(X)$. Given that $\{r_1, \ldots, r_N\}$ is a finite set, we may study its *permutations*, i.e., one-to-one mappings of this set. The set of all such permutations forms a group, known as the *symmetric group* S_N.[19] Now, the roots may jointly satisfy an algebraic relationship, say $h(r_1, \ldots, r_N) = 0$, where $h(Y_1, \ldots, Y_N)$ is a multivariate polynomial with rational coefficients. The *Galois group* of $g(X)$, denoted by $\mathcal{G}$, consists of those permutations π of the roots with the property that any algebraic relationship $h(r_1, \ldots, r_N) = 0$ satisfied by the roots remains satisfied after applying π to the roots, i.e., $h(r_{\pi(1)}, \ldots, r_{\pi(N)}) = 0$. We illustrate the concept with an example.

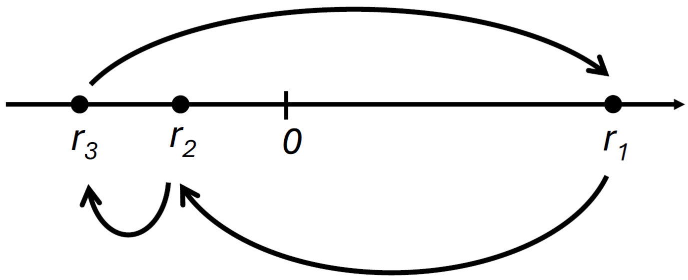

Figure 1: The permutation π_0 operates on the roots of the polynomial $g_0(X)$.

software instead. See Appendix A.7 for details.

[19]In abstract algebra, a finite *group* consists of a finite set of elements (here: a set of permutations of the roots of a polynomial), a binary operation (here: the composition of permutations), and a unit element (here: the trivial permutation that keeps all roots fixed) such that (i) the group operation is associative, (ii) every element has an inverse, and (iii) the group operation with the unit element has no effect. E.g., in $S_3 = \{(), (12), (13), (23), (123), (132)\}$, the neutral element is $()$, and the permutation (123) maps roots r_1 to r_2, r_2 to r_3, and r_3 to r_1. Moreover, the group operation corresponds to the execution of the permutations from right to left. E.g., in S_3, we have $(12)(123) = (23)$. It is important to realize that the group operation need not be commutative. And indeed, $(123)(12) = (13) \neq (23)$.

Example 2 *The polynomial* $g_0(X) = X^3 - 9X - 9$ *has the three real roots* $r_1 = 3.41$, $r_2 = -1.18$, *and* $r_3 = -2.23$, *as illustrated in Figure 1.*[20] *The Galois group* $\mathcal{G}_0$ *of* $g_0(X)$ *turns out to be the cyclic group of order three,*[21] *a generator of which is the permutation* $\pi_0 = (123)$ *that maps* r_ν *to* $r_{\nu+1}$ *if* $\nu \in \{1, 2\}$ *and* r_3 *to* r_1. *To understand why some permutations are members of the Galois group, while others are not, consider the following two illustrations. First, the algebraic relationship* $(r_1 - r_2)(r_2 - r_3)(r_1 - r_3) = 27$ *remains valid if* π_0 *is applied to the roots. One can show that this is actually true for any algebraic relationship in the three variables* r_1, r_2, *and* r_3 *with rational coefficients. This means by definition that* π_0 *is a member of* $\mathcal{G}_0$. *Second, the very same relationship becomes invalid if, for instance, the permutation* $\pi_1 = (12)$ *is applied, because*

$$(r_{\pi_1(1)} - r_{\pi_1(2)})(r_{\pi_1(2)} - r_{\pi_1(3)})(r_{\pi_1(1)} - r_{\pi_1(3)}) = (r_2 - r_1)(r_1 - r_3)(r_2 - r_3) = -27 \neq 27.$$

Thus, π_1 *is not a member of* $\mathcal{G}_0$.

2.3 Galois' equivalence[22]

The gist of Galois theory is that the knowledge of the Galois group of an irreducible polynomial allows to decide if the correponding polynomial equation can be solved by radicals or not. In particular, for a polynomial $g(X)$ of degree five that is irreducible over the rationals, it suffices to show that the Galois group is, e.g., the full symmetric group S_5 to be able to deduce that $g(X) = 0$ cannot be solved by radicals.

[20] This polynomial happens to arise in the analysis of a Tullock contest with $n = 3$, $(V_1, V_2, V_3) = (6, 3, 2)$, and $R = \frac{1}{2}$. See Appendix A.4.

[21] The *order* of a finite group is the number of its elements.

[22] The main result of Galois theory (over the field of rational numbers) is the inclusion-reversing isomorphism of the respective lattices of (i) subfields of a field extension K over $\mathbb{Q}$, and (ii) subgroups of the Galois group associate with K. That result, known as Galois' correspondence, is used to derive Galois' equivalence (cf. Lemma 1 below).

By definition, the group $\mathcal{G}$ is *solvable* if there is a finite sequence of subgroups

$$\{1\} = \mathcal{G}^{(0)} \subsetneq \mathcal{G}^{(1)} \subsetneq \cdots \subsetneq \mathcal{G}^{(K)} = \mathcal{G}, \tag{2}$$

such that each $\mathcal{G}^{(k)}$ is a normal subgroup of $\mathcal{G}^{(k+1)}$, and all quotient groups $\mathcal{G}^{(k+1)}/\mathcal{G}^{(k)}$ are abelian.[23] Intuitively, a non-abelian quotient group in a maximally refined sequence (2) is a "smoking gun" for the existence of a root that cannot be found by basic arithmetic operations and the extraction of roots alone.

Lemma 1 (Galois) *Let $g(X)$ be an irreducible polynomial. Then, the following statements are equivalent:*

(i) $g(X) = 0$ is solvable by radicals;

(ii) the Galois group of $g(X)$ is solvable.

Proof. See Stewart (2015, Thms. 15.8 and 18.21). □

We illustrate Lemma 1 by returning to our earlier example.

Example 2 (continued) *The cyclic group of order three, $\mathbb{Z}/3\mathbb{Z} \subseteq S_3$, which is generated by the permutation $\pi_0 = (123)$, admits the trivial decomposition*

$$\begin{array}{ccc} \mathcal{G}^{(0)} & \subseteq & \mathcal{G}^{(1)} \\ \shortparallel & & \shortparallel \\ \{1\} & \subseteq & \mathbb{Z}/3\mathbb{Z}. \end{array}$$

Thus, $\mathbb{Z}/3\mathbb{Z}$ is solvable. By Lemma 1, this is equivalent to the statement that the equation $g_0(X) = 0$ is solvable by radicals. And indeed, an application of Cardano's

[23] A subgroup $\mathcal{N}$ of a group $\mathcal{G}$ is called *normal* if $\pi\mathcal{N}\pi^{-1} = \mathcal{N}$ for all elements $\pi \in \mathcal{G}$. It is a basic result in group theory that the cosets $\pi\mathcal{N}$ form a group, known as the *quotient group*. A group $\mathcal{G}$ is called *abelian* if the group operation is commutative.

formula yields $r_{1,2,3} = \sqrt[3]{\frac{9+\sqrt{-27}}{2}} + \sqrt[3]{\frac{9-\sqrt{-27}}{2}}$*, where the three solutions result from choosing different values for the cubic roots (see Appendix A.3). Thus, the solutions of* $g_0(X) = 0$ *may indeed be represented using basic arithmetic operations and the extraction of roots alone.*[24]

In the example, we knew of course about the existence of an explicit formula before, because the degree of $g_0(X)$ is just three, but the point is that Lemma 1 holds for polynomials of any degree.

3 Tractable cases of the Tullock contest

In this section, we review the main results in the literature that provide an explicit solution of the Tullock contest in special cases. For convenience, we will restrict attention to equilibria in pure strategies and to cases in which the equilibrium is unique.

3.1 Homogeneous valuations

Suppose first that $V_1 = \ldots = V_n \equiv V > 0$, i.e., all contestants possess the same positive valuation of winning. Assuming that the solution is symmetric, so that $x_1 = \ldots = x_n$, the necessary first-order condition simplifies and yields equilibrium efforts of

$$x_i^* = \frac{(n-1)RV}{n^2}. \tag{3}$$

Clearly, each contestant wins with equal probability $p_i^* = \frac{1}{n}$. The corresponding equilibrium payoffs are, therefore, given by

$$\Pi_i^* = \frac{(n-(n-1)R)V}{n^2}. \tag{4}$$

[24]Contrary to what one might expect, the explicit representation in this and similar examples may require (i) extracting roots from complex numbers even if the polynomial has only real roots (van der Waerden, 1931, pp. 188-189), and (ii) taking repeated radicals even if the extension is cyclic and of prime degree (Kang, 2000, Thm. 1).

A closer inspection of the equilibrium conditions shows that these formulas indeed characterize the unique Nash equilibrium in pure strategies if $R \leq \frac{n}{n-1}$.[25]

Proposition 1 *Suppose that $V_1 = \ldots = V_n \equiv V$ and that $R \leq \frac{n}{n-1}$. Then, the pure-strategy Nash equilibrium is unique, symmetric, interior, and characterized by (3) and (4).*

Proof. For $R \leq 1$, the claim follows from Pérez-Castrillo and Verdier (1992, Prop. 4a). Suppose next that $R > 1$. In that case, our condition that $R \leq \frac{n}{n-1}$ is equivalent to $n \leq \frac{R}{R-1}$. Under that latter condition, however, the equilibrium property as well as uniqueness has been established by Pérez-Castrillo and Verdier (1992, Prop. 4b). □

3.2 Constant returns

Suppose next that $R = 1$. Again, contestant i's necessary first-order condition simplifies and turns into

$$\frac{X - x_i}{X^2} V_i - 1 = 0,$$

where $X = x_1 + \ldots + x_n$ denotes the aggregate effort level. Hence, individual efforts, winning probabilities, and payoffs may be computed from Σ via the formulas

$$x_i = X\left(1 - \frac{X}{V_i}\right), \tag{5}$$

$$p_i = 1 - \frac{X}{V_i}, \tag{6}$$

$$\Pi_i = \left(1 - \frac{X}{V_i}\right)^2 V_i. \tag{7}$$

[25] For $R \in (\frac{n}{n-1}, 2]$, there are multiple asymmetric equilibria in pure strategies. In that case, which obviously requires $n \geq 3$, the number of contestants $\widetilde{n}$ exerting a positive effort may vary, but the equilibrium efforts and payoffs of those active contestants are characterized as above with n replaced by $\widetilde{n}$. For $R > 2$, there is no equilibrium in pure strategies. See Cornes and Hartley (2005, Thm. 7) and Ryvkin (2007, Sec. 3).

Adding the n equations for the effort and rewriting yields

$$X = \frac{(n-1)\overline{V}_n}{n}, \tag{8}$$

where $\overline{V}_n = n\left(V_1^{-1} + \ldots + V_n^{-1}\right)^{-1}$ denotes the harmonic mean of contestants' valuations. An inspection of the optimality of entry shows that this indeed characterizes an interior Nash equilibrium in pure strategies if $V_n > \frac{n-1}{n}\overline{V}_n$, or equivalently, if $V_n > \frac{n-2}{n-1}\overline{V}_{n-1}$.

Proposition 2 *Suppose that $R = 1$ and $V_n > \frac{n-2}{n-1}\overline{V}_{n-1}$. Then, the pure-strategy Nash equilibrium is unique, interior, and characterized by (5)-(8).*

Proof. The equilibrium property has been established by Hillman and Riley (1989, Prop. 5). The observation that the equilibrium is unique has been made by Stein (2002, Prop. 1).[26] □

If the condition on the valuations is not satisfied, then contestant n exerts zero effort, and the characterization applies analogously with $(n-1)$ replacing n. Proceeding recursively, the set of contestants exerting a positive effort in the likewise unique equilibrium in which some of the contestants exert zero effort can be easily identified.

3.3 Two active contestants

Suppose that $n = 2$. In this case, the two first-order conditions read

$$\begin{aligned} \frac{Rx_1^{R-1}x_2^R}{(x_1^R + x_2^R)^2}V_1 &= 1, \\ \frac{Rx_2^{R-1}x_1^R}{(x_1^R + x_2^R)^2}V_2 &= 1. \end{aligned}$$

[26] See also Matros (2006).

Dividing the two equations yields

$$x_2 = \lambda x_1,$$

with $\lambda = V_2/V_1$, which allows eliminating the unknown x_2 in the first-order condition of contestant 1. Rearranging, one arrives at effort levels

$$x_i^* = \frac{R\lambda^R}{(1+\lambda^R)^2} V_i. \tag{9}$$

From this, we obtain winning probabilities $p_i^* = V_i/(V_1 + V_2)$, as well as payoffs

$$\Pi_1^* = \frac{1+(1-R)\lambda^R}{(1+\lambda^R)^2} V_1 \tag{10}$$

$$\Pi_2^* = \frac{\lambda^{2R}+(1-R)\lambda^R}{(1+\lambda^R)^2} V_2. \tag{11}$$

It can now be checked that those equations characterize the unique Nash equilibrium in pure strategies if $R \leq 1 + \lambda^R$.[27]

Proposition 3 *Suppose that $n = 2$ and that $R \leq 1 + \lambda^R$. Then, the pure-strategy Nash equilibrium is unique, interior, and characterized by (9)-(11).*[28]

Proof. See Nti (1999). □

Apart from the cases surveyed above, we are not aware of any explicit characterizations of the pure-strategy Nash equilibrium in the literature. It is known, however, that the solution is unique and interior in the case $R \in (0, 1)$ regardless of $V_1 \geq \ldots \geq V_n > 0$ (Szidarovszky and Okuguchi, 1997). We will therefore focus on that case in the sequel.

[27]For $n = 2$ and $R > 1 + \lambda^R$, there are no equilibria in pure strategies. If $R \in (1 + \lambda^R, 2]$, then contestant 1 uses a pure strategy while contestant 2 uses a mixed strategy (this case obviously requires $V_2 < V_1$). See Ewerhart (2017b) and Feng and Lu (2017). For $R > 2$, the equilibrium is in mixed strategies (Baye et al., 1994; Alcalde and Dahm, 2010; Ewerhart, 2015, 2017a).

[28]It apparently went unnoticed in the literature that, in the case of strictly increasing returns to scale, Proposition 3 admits a straightforward extension to the case of $n \geq 3$ contestants. For details, see Appendix A.2.

4 A limitation of tractability

This section documents our main contribution. We start by providing a formal definition of tractability (Subsection 4.1), then present an example (Subsection 4.2), and finally, derive the main result of the present paper (Subsection 4.3).

4.1 A formal definition of tractability

We fix $n \geq 2$ and $R > 0$. By the *primitives of the model*, we mean the exogenous parameters $V_1 \geq \ldots \geq V_n > 0$. We define tractability as follows.

Definition 1 *We say that the n-player Tullock contest with parameter $R > 0$ is* ***solvable by radicals*** *if its key endogenous characteristics, including individual equilibrium efforts, winning probabilities, and payoffs, may be determined from the primitives of the model by repeatedly taking sums, differences, products, and quotients, as well as extracting roots.*[29]

For the applied economist, the definition might seem restrictive at first sight. E.g., additional functions could be added to the set of admissible operations. However, as will be discussed below, there are good reasons to assume that the definition captures, in the considered class of games, what is intuitively understood by tractability. Definition 1 also seems to be the first formal definition of algebraic tractability in the economics literature, i.e., it might be of independent interest.

It should be noted that the tractable cases surveyed above all satisfy the definition. A particular case is Proposition 3, which exhibits the term $\lambda^R = (V_2/V_1)^R$ in the

[29]If the contest admits multiple equilibria, one might want to require that *all* equilibria may be characterized explicitly. However, this question will be of minor relevance in the sequel as all our examples feature equilibrium uniqueness.

solution formulas. If $R = \frac{p}{q}$ is a rational number, however, then $\lambda^R = \lambda^{p/q} = \sqrt[q]{\lambda^p}$ is representable using radicals. Thus, given that the set of rational numbers is dense in the reals, there is little that is lost.

Our main result (Theorem 1 below) says that the n-player Tullock contest is, in general, not solvable by radicals. It will even show that none of the individual characteristics, viz. effort levels, winning probabilities, and payoffs, can be determined using basic arithmetic operations and the extraction of roots alone.

4.2 An intractable example

The following example illustrates that, beyond the cases surveyed in the previous section, it may be impossible to solve the n-player Tullock contest using radicals.

Example 3 *Suppose that there are $n = 5$ contestants, with valuations*

$$(V_1, V_2, V_3, V_4, V_5) = (5, 4, 3, 2, 1).$$

Suppose, in addition, that $R = \frac{1}{2}$. Then, there is a unique Nash equilibrium in pure strategies

$$\begin{aligned} x^* &= (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) \\ &= (0.53, 0.38, 0.25, 0.13, 0.04), \end{aligned}$$

but for none of the players, equilibrium efforts nor winning probabilities nor payoffs can be derived from the primitives of the model using basic arithmetic operations and the extraction of roots alone.

Next, we explain why Example 3 is not tractable. For $R = \frac{1}{2}$, but still general $n \geq 2$ and general valuations, the necessary first-order conditions may be rewritten as

$$\frac{\sqrt{X} - \sqrt{x_i}}{2\sqrt{x_i}X} V_i - 1 = 0,$$

where $X = (\sqrt{x_1} + \ldots + \sqrt{x_n})^2$ is a *generalized aggregate* of individual efforts. One notes that these conditions are also sufficient because the marginal return is infinite at the zero effort level, so that all contestants choose a positive effort.[30] Solving for $\sqrt{x_i}$ yields

$$\sqrt{x_i} = \frac{\sqrt{X} V_i}{2X + V_i}.$$

Summing over all contestants, and subsequently dividing by $\sqrt{X} > 0$, one obtains the key equation

$$\frac{V_1}{2X + V_1} + \ldots + \frac{V_n}{2X + V_n} = 1. \tag{12}$$

Given that the left-hand side assumes the value n for $X = 0$ and is continuously diminishing in X, this equation uniquely characterizes the equilibrium value $X^* > 0$. Moreover, given X^*, one obtains

$$x_i^* = \frac{X^* V_i^2}{(2X^* + V_i)^2}, \tag{13}$$

$$p_i^* = \frac{\sqrt{X^*} V_i^2}{(2X^* + V_i)^2}, \tag{14}$$

$$\Pi_i^* = \frac{\sqrt{X^*}(V_i - \sqrt{X^*}) V_i^2}{(2X^* + V_i)^2}. \tag{15}$$

Conversely, if for some $i \in \{1, \ldots, n\}$, just one of the endogenous characteristics x_i^*, p_i^*, or Π_i^* could be expressed in explicit terms, then we could easily derive the value of X^* by solving quadratic equations.

From the above, the generalized aggregate X in our example satisfies

$$\frac{5}{2X+5} + \frac{4}{2X+4} + \frac{3}{2X+3} + \frac{2}{2X+2} + \frac{1}{2X+1} = 1.$$

[30] Indeed, using a suitable substitution (e.g., Szidarovszky and Okuguchi, 1997), this case may be rephrased as a Tullock contest with constant returns and quadratic costs.

Multiplying through and collecting terms, one finds that X^* is the unique positive root of the polynomial

$$g_5(X) = 8X^5 - 170X^3 - 450X^2 - 411X - 120. \tag{16}$$

Ignoring negative solutions,[31] one numerically finds the root $X = 5.72$.

Our discussion so far may be summarized as follows.

Lemma 2 *The following statements are equivalent:*

(i) For some player i, the equilibrium effort x_i^* is solvable by radicals;[32]

(ii) for some player i, the equilibrium winning probability p_i^* is solvable by radicals;

(iii) for some player i, the equilibrium payoff Π_i^* is solvable by radicals;

(iv) $g_5(X) = 0$ is solvable by radicals.

Proof. See the text above. □

To establish that Example 3 is not tractable, we will show that $g_5(X)$ is not solvable by radicals. In view of Lemma 1, it suffices to check the following facts.

Lemma 3 *The following statements are true:*

(i) $g_5(X)$ is irreducible;

(ii) the Galois group of $g_5(X)$ is the full symmetric group S_5;

[31] Indeed, in addition to the unique positive solution, equation (12) has a total of $(n-1)$ negative solutions, each of which is strictly located between some pair of neighboring poles.

[32] Note the abuse of the terminology. We mean here that x_i^*, etc., may be expressed by repeatedly applying basic arithmetic operations and roots to rational numbers.

(iii) S_5 is not solvable.

Proof. For convenience, parts (i) and (ii) have been established using the software tool *Sage*. See Appendix A.7 for details. (iii) The fact that S_5 is not solvable is well-known. See, e.g., Stewart (2015, Cor. 14.8). $\square$

Thus, we may conclude that Example 3 is not solvable by radicals.

4.3 A general result

We arrive at the main result of this paper.

Theorem 1 *The n-player Tullock contest with parameter $R > 0$ is, in general, not solvable by radicals.*

Proof. See Appendix A.5. $\square$

Theorem 1 is obtained by contradiction. If a formula existed for the equilibrium in the n-player contest, given valuations $V_1 \geq \ldots \geq V_n > 0$ and a parameter $R > 0$, possibly with case distinctions, then it should specialize, in particular, to a formula in the case $R = \frac{1}{2}$. In the Appendix, we determine the general form of equation (16) as

$$g_n(X; V_1, \ldots, V_n) = \sum_{k=0}^{n}(1 - k)\sigma_k(V_1, \ldots, V_n)(2X)^{n-k},$$

where $\sigma_k(V_1, \ldots, V_n)$ denotes the *elementary symmetric polynomial* of degree k in the variables $V_1, \ldots, V_n$.[33] As discussed above, with economically meaningful choices of the valuation vector, $g_n(X; V_1, \ldots, V_n)$ has n distinct roots of multiplicity one, and the generalized aggregate $X^* = X^*(V_1, \ldots, V_n)$ as its sole positive root. We show that

[33] Thus, $\sigma_0 = 1$, $\sigma_1 = V_1 + \ldots + V_n$, $\sigma_2 = V_1V_2 + \ldots + V_{n-1}V_n$, and so on, up to $\sigma_n = V_1 \cdot \ldots \cdot V_n$. Cf. Stewart (2015, Sect. 18.2).

the Galois group of $g_n(X; V_1, \ldots, V_n)$ over the function field $\mathbb{Q}(V_1, \ldots, V_n)$ contains the Galois group of $g_{n-1}(X; V_1, \ldots, V_{n-1})$ as a subgroup. That result is obtained using the concept of *specialization* in Galois theory (van der Waerden, 1931, §61). Intuitively, we replace V_n by the value zero, which amounts to a well-defined mapping from the integral domain of parameterized polynomials $\mathbb{Q}(V_1, \ldots, V_n)[X]$ onto its subring $\mathbb{Q}(V_1, \ldots, V_{n-1})[X]$, even though we cannot do this in any solution formula.[34] Hence, by induction, we can make our way down to $g_5(X; V_1, \ldots, V_5)$. The Galois group of that polynomial over $\mathbb{Q}(V_1, \ldots, V_5)$, however, can be shown to be the full symmetric group S_5, because it specializes to Example 3. As S_5 is not solvable, we may use Lemma 1 to deduce that $g_n(X; V_1, \ldots, V_n)$ is not solvable by radicals over $\mathbb{Q}(V_1, \ldots, V_n)$.

5 Discussion

This section offers some discussion. We first argue that, for the considered class of games, our concept of tractability is unlikely to be affected if we were to add additional elementary functions as admissible operations (Subsection 5.1). Next, we point out that, whatever definition we ultimately choose, there will always be a need for a postulate that the preferred definition captures the intuitive notion of tractability (Subsection 5.2). Finally, we explain why explicit solutions can also not be expected for values of the parameter R other than $\frac{1}{2}$ (Subsection 5.3).

5.1 Adding elementary functions

As noted before, one might argue that the notion of tractability promoted in this study is of limited interest because economists are familiar with a number of elementary

[34]Indeed, V_n might occur in the denominator of the solution formula, so that setting $V_n = 0$ may not be a well-defined operation.

functions, such as the exponential, the logarithm, and various trigonometric functions, for instance. Upon reflection, however, it seems very unlikely that the addition of elementary functions would resolve the tractability issue for the Tullock contest. After all, this would imply, for instance, that the quintic constructed in the discussion of Example 3 is tractable by allowing for elementary functions. But the properties of unsolvable quintics have been studied very thoroughly by mathematicians for more than two centuries, making this indeed a very remote possibility. Thus, even if our definition of tractability excludes the use of elementary functions, it seems to capture quite accurately which polynomial equations are tractable in a practical sense and which are not.[35]

5.2 The need for a postulate

There is, however, another problem that cannot be wiped away so easily, viz. that the tractability of a model is a rather vague concept. The situation is reminiscent of the problem of defining effective computability in computer science. It requires a *postulate*, such as the Church-Turing thesis, to assert that computability in the intuitive sense (i.e., by a scientist) is equivalent to computability in the formal sense (e.g., by a Turing machine). Similarly here, it seems to require a postulate to assert that tractability in the practical sense is equivalent to tractability in the formal sense.[36]

[35]The mathematical literature has identified ways to deal with the implications of Galois' theory. According to a survey poster (Wolfram Research, 2005), there are solution approaches for general polynomial equations based on continued fractions, modular forms, theta functions, infinite series representations, Mellin integrals, hypergeometric functions, and elliptic Siegel functions. Such approaches, however, tend to add a multivariate transcendental function to the set of admissible operations (Umemura, 2007). The extent to which those methods are suitable for economic analysis is, therefore, still to be explored.

[36]Conversely, one might also argue that Definition 1 is too generous in some cases, e.g., if the resulting formulas are difficult to interpret or to work with. Given the purpose of the present paper, however, which is an impossibility result, we are on the safe side with respect to that concern.

5.3 Alternative values of R

Theorem 1 is a partial result in the sense that there might exist values of the parameter R different from $\frac{1}{2}$ for which a solution formula exists. However, that possibility is likewise unlikely. In Appendix A.6, we show that, for $R = \frac{p}{q}$, with (relatively prime) integers $q > p > 0$, the unique vector of equilibrium efforts solves a system of algebraic equations. As has been seen, for $p = 1$ and $q = 2$, that system can be reduced by hand to a single polynomial equation for the generalized aggregate X. For other values of R, however, that simplification is not available. As a result, the univariate polynomial in the Gröbner basis becomes more complex. Table I illustrates this fact by reporting the degree d of the univariate polynomial in the Gröbner basis produced by *Mathematica* using the default monomial order in a three-player Tullock contest with valuations $(V_1, V_2, V_3) = (3, 2, 1)$.[37] In some cases, the degree could be further lowered using a suitable substitution. E.g., x_3^2 is replaced by y_3, x_3^4 by y_3^2, etc. For those cases, the lowered degree d' is shown as well.

p/q	1/2	1/3	2/3	1/4	3/4	1/5	2/5	3/5	4/5
d	6	21	19	46	40	83	93	83	69
d'	3	7	-	-	20	-	-	-	-

Table I: Degrees of the univariate polynomial in a Gröbner basis

The only other case in which we were able to determine a Galois group was $p = 1$ and $q = 3$, corresponding to $R = \frac{1}{3}$. In that case, the polynomial was of degree $d = 7$, irreducible, and of Galois group S_7, which is unsolvable. In all other cases, the degree of the univariate polynomial in the Gröbner basis is so large that, even if an example

[37] We work with specific values here because the parameterized Gröbner basis had several million terms.

could be found in which the Galois group is solvable, the resulting explicit formula either might never be found or, if it can be found, would be of little practical value.[38]

6 Concluding remarks

In this paper, we have shown that the pure-strategy Nash equilibrium of the n-player Tullock contest cannot, in general, be expressed in terms of the primitives of the model using basic arithmetic operations and the extraction of roots alone. We have also explained why the issue cannot be easily resolved by adding familiar functions such as the exponential or the logarithm. Thus, the analysis clearly delineates the boundaries of tractability for the Tullock model.

What is the scope of the methods developed in this paper? We believe that the formal definition of tractability should be applicable in other contexts as well. Indeed, in numerous economic models such as Bertrand pricing games, Walrasian exchange economies, and arms races with incomplete information, equilibria may be characterized as solutions of systems of polynomial equations, possibly involving additional inequalities (Judd et al., 2012). Applications of the shape lemma, followed by an analysis in line with what has been accomplished in the present paper, might then allow to decide, once and for all, if such models are solvable by radicals or not. Like in the case of the Tullock contest dealt with in the present paper, this might help to end speculations about possible generalizations of important classes of economic models.

[38]The story that might come to one's mind is that, at the end of the 19th century, the German mathematician Johann Gustav Hermes spent a decade to showcase the construction of the regular 65537-gon with straightedge and compass.

A Appendix

This appendix contains material that has been omitted from the body of the paper.

A.1 Details on Example 1

We first check the equilibrium property directly. Given that $R < 1$, it is never optimal for any contestant to exert an effort of zero. Moreover, payoff functions are globally strictly concave in own effort in the interior. It therefore suffices to check the first-order conditions. Indeed, define the generalized aggregate

$$\begin{aligned} X^* &= (\sqrt{x_1^*} + \sqrt{x_2^*} + \sqrt{x_3^*} + \sqrt{x_4^*} + \sqrt{x_5^*})^2 \\ &= \left(\sqrt{\tfrac{1944}{625}} + \sqrt{\tfrac{216}{121}} + \sqrt{\tfrac{24}{25}} + \sqrt{\tfrac{1176}{3025}} + \sqrt{\tfrac{24}{625}} \right)^2 \\ &= 24. \end{aligned}$$

One may now check mechanically that the claimed value for each x_i^* satisfies the first-order condition

$$\frac{\sqrt{X^*} - \sqrt{x_i^*}}{2X^* \sqrt{x_i^*}} V_i - 1 = 0.$$

This shows that we have indeed identified an equilibrium.

To understand why this special case is tractable, we aggregate the five first-order conditions into the single polynomial equation (cf. the details on Example 3 provided in the body of the paper)

$$\hat{g}(X) = 4X^5 - 1553X^3 - 15864X^2 - 50139X - 40824 = 0.$$

But $\hat{g}(X)$ fails to be irreducible. In fact, one can check that

$$\hat{g}(X) = (X - 24)\,(4X^4 + 96X^3 + 751X^2 + 2160X + 1701),$$

so that $X^* = 24$ is a solution for the generalized aggregate.

A.2 Extension of Proposition 3

We did not find a reference for the following result.

Proposition A.1 *Suppose that $V_1 \geq \ldots \geq V_n > 0$ with $n \geq 3$. Suppose also that $R \in [1, 1 + \lambda^R]$, and that*

$$V_3 \leq V_1 \frac{\lambda^R}{\left(1+\lambda^R\right)^{2-1/R}} \frac{R^2}{(R-1)^{1-1/R}}. \tag{A.1}$$

Then, the n-player Tullock contest admits a pure-strategy Nash equilibrium characterized by (9)-(11) for contestants $i \in \{1, 2\}$, and by $x_i^* = p_i^* = \Pi_i^* = 0$ for contestants $i \in \{3, \ldots, n\}$.

Proof. It follows from Proposition 3 that contestants 1 and 2 play a best response. To establish that none of the contestants $i \in \{3, \ldots, n\}$ have an incentive to deviate, it certainly suffices to consider contestant $i = 3$. By Cornes and Hartley (2005, Prop. 4), the zero bid is a best response for contestant 3 given effort levels $x_1 = x_1^*$, $x_2 = x_2^*$, and $x_j = 0$ for any $j \geq 4$, if and only if

$$\left(x_1^*\right)^R + \left(x_2^*\right)^R \geq V_3^R \frac{(R-1)^{R-1}}{R^R}.$$

Using (9), this transforms into (A.1). $\square$

In the constant-returns case $R = 1$, we have $(R-1)^{1-1/R} = 0^0 = 1$, so that inequality (A.1) becomes $V_3 \leq \frac{1}{2}\overline{V}_2$. Thus, this simply brings us back to the case dealt with in Proposition 2.

The equilibrium identified in Proposition A.1 need not be unique, however. To see this, it suffices to consider the case where $R = 1 + \lambda^R$. Then, inequality (A.1) reads $V_3 \leq V_2 R^{1/R}$, which is automatically fulfilled. Thus, by continuity, for R close to $1+\lambda^R$

and valuations close to identical, there are multiple pure-strategy equilibria, viz. one for each pair of active contestants.

A.3 The principal value of the N-th root

This section provides further background regarding the N-th root. It also prepares the analysis of the case $N = 3$ and $R = \frac{1}{2}$ dealt with in the next section. As discussed in the body of the paper, the N-th root $\sqrt[N]{z}$ of a complex number $z \neq 0$ is defined only up to a unit-root factor. Thus, any $z \neq 0$ admits precisely N pre-images that differ by powers of the N-th unit root $\zeta_N = \exp(2\pi\sqrt{-1}/N)$. To resolve the resulting ambiguity, one may refer to the *principal value* of $\sqrt[N]{z}$, which is defined as follows. Given $z \neq 0$, we find unique polar coordinates $r > 0$ and $\varphi \in (-\pi, \pi]$ such that $z = r\exp(\varphi\sqrt{-1})$. Then, using de Moivre's identity,

$$\sqrt[N]{z} = \sqrt[N]{r}\exp\left(\frac{\varphi}{N}\sqrt{-1}\right)$$

is a N-th root of z, known as the principal value of the N-th root.

A special case of interest is a root of a complex number with positive real part.

Lemma A.1 *Let $z = x + y\sqrt{-1}$, with $x > 0$ and y real. In this case, the principal value of the N-root of z is given by*

$$\sqrt[N]{z} = \sqrt[2N]{x^2 + y^2}\left(\cos(\frac{1}{N}\arctan(\frac{y}{x})) + \sqrt{-1}\sin(\frac{1}{N}\arctan(\frac{y}{x}))\right).$$

Proof. In the considered case, $r = \sqrt{x^2 + y^2}$ *and* $\varphi = \arctan(y/x)$. Hence, the principal value is given as

$$\sqrt[N]{z} = \sqrt[2N]{x^2 + y^2}\exp\left(\frac{\sqrt{-1}}{N}\arctan(\frac{y}{x})\right).$$

The claim follows now from Euler's formula $\exp(\varphi\sqrt{-1}) = \cos\varphi + \sqrt{-1}\sin\varphi$. □

A.4 The case $n = 3$ and $R = \frac{1}{2}$

It follows from the present analysis that the case $R = \frac{1}{2}$ is tractable for any $n \geq 2$, provided that valuations are taken from at most four different values. For example, the Tullock contest with $n = 5$ players, where $V_1 = V_2 > V_3 > V_4 > V_5 > 0$ and $R = \frac{1}{2}$, admits an explicit solution, because the analysis of first-order conditions leads to a polynomial equation of degree four, which can always be solved using radicals (van der Waerden, 1931, §58). Below, we derive the explicit solution for $n = 3$.

Proposition A.2 *Suppose that there are $n = 3$ contestants, with valuations $V_1 \geq V_2 \geq V_3 > 0$. Suppose also that $R = \frac{1}{2}$. Then, the solution of the Tullock contest is given by equations (13-15) in the body of the paper, where*

$$X^* = \frac{1}{\sqrt{c_a c_g}} \cdot \cos\left(\frac{1}{3}\arctan\left\{\left(\frac{c_a}{c_g}\right)^3 - 1\right\}\right), \tag{A.2}$$

and where c_a and c_g denote, respectively, the arithmetic and geometric means of the reciprocal valuations $\frac{1}{V_1}$, $\frac{1}{V_2}$, and $\frac{1}{V_3}$.

Proof. Suppose first that valuations are homogeneous, i.e., $V_1 = V_2 = V_3 \equiv V$. Then, from Proposition 1, $x_1^* = x_2^* = x_2^* = \frac{V}{9}$. Hence,

$$X^* = \left(\sqrt{x_1^*} + \sqrt{x_2^*} + \sqrt{x_3^*}\right)^2 = V.$$

But equation (A.2) delivers the same result because $c_a = c_g = \frac{1}{V}$ in that case. This proves the claim in the case of homogeneous valuations. Suppose next that not all valuations are identical. Then, $c_g < c_a$ (Hardy et al., 1934, p. 17). Under the assumptions made, equation (12) reads

$$\frac{V_1}{2X + V_1} + \frac{V_2}{2X + V_2} + \frac{V_3}{2X + V_3} = 1.$$

Multiplying through with the common denominator and collecting terms, we obtain a depressed[39] cubic equation

$$X^3 + pX + q = 0, \tag{A.3}$$

where $p = -3c_a/c_g^3$ and $q = -2/c_g^3$. The *discriminant* is

$$D = -4p^3 - 27q^2 = \frac{2^2 \cdot 3^3}{c_g^6} \left(\left(\frac{c_a}{c_g} \right)^3 - 1 \right) > 0.$$

Thus, the cubic has three real solutions, one of which is positive and two of which are negative. We are in the *casus irreducibilis*, i.e., any explicit solution by radicals requires the extraction of roots from complex numbers. The solution is $X^* = C - \frac{p}{3C}$, with a total of six possibilities for

$$C = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} = \frac{1}{c_g} \sqrt[3]{1 \pm \sqrt{1 - (c_a/c_g)^3}}.$$

Hence,

$$X^* = \frac{1}{c_g} \left(\sqrt[3]{1 \pm \sqrt{1 - (c_a/c_g)^3}} + \frac{(c_a/c_g)}{\sqrt[3]{1 \pm \sqrt{1 - (c_a/c_g)^3}}} \right), \tag{A.4}$$

where the $\pm$ in front of the square roots assume the same value and the cubic roots take the same value. To select the values of the roots, one notes first that the right-hand side of equation (A.4) does not depend on the sign in front of the square root. Indeed, regardless of the choice of the cubic root, the first term in the large brackets can be easily seen to be the complex conjugate of the second term. We may therefore, without loss of generality, select the positive sign in front of the square root. Further, we know that X^* is the unique positive solution of equation (A.3). But the only way to arrive

[39] A cubic equation is called *depressed* if there is no quadratic term.

at a positive value in equation (A.4) is to select the principal value for the cubic root. It now suffices to apply Lemma A.1 to obtain the claimed formula for X^*. $\square$

Equation (A.2) uses trigonometric functions to circumvent the extraction of roots from complex numbers. This approach goes back to Viète, who proposed it to avoid the use of complex numbers in the casus irreducibilis of Cardano's analysis (cf. Plante, 2018).[40]

We illustrate the use of Proposition A.2 with an example.

Example A.1 *Suppose that* $(V_1, V_2, V_3) = (8, 2, 1)$. *Then,* $c_a = \frac{1}{3}(\frac{1}{V_1} + \frac{1}{V_2} + \frac{1}{V_3}) = \frac{13}{24} = 0.54$ *and* $c_g = 1/\sqrt[3]{V_1 V_2 V_3} = \frac{1}{4} = 0.25$. *Hence,* $X^* = 2.401$. *Individual efforts are, therefore, given by* $x_1^* = 0.94$, $x_2^* = 0.21$, *and* $x_3^* = 0.07$. *Winning probabilities are* $p_1^* = 0.57$, $p_2^* = 0.27$, *and* $p_3^* = 0.16$. *Payoffs are* $\Pi_1^* = 3.64$, $\Pi_2^* = 0.33$, *and* $\Pi_3^* = 0.09$.

In the case captured by Proposition A.2, the discriminant D is known to determine the Galois group. Specifically, if D is not a square of a rational number, then the Galois group is S_3. An example is $(V_1, V_2, V_3) = (3, 2, 1)$. Then, $D = 2^2 \cdot 359$ is not a square. And indeed, the corresponding polynomial $g(X) = 4X^3 - 11X - 6$ is irreducible with Galois group S_3. If, however, D happens to be square, then that Galois group is the cyclic group of order three, i.e., $\mathbb{Z}/3\mathbb{Z}$. An example is $(V_1, V_2, V_3) = (6, 3, 2)$. Then, $D = 2^6 \cdot 3^6$ is a square. And indeed, the corresponding polynomial $g(X) = X^3 - 9X - 9$ is irreducible with cyclic Galois group $\mathbb{Z}/3\mathbb{Z}$ (cf. Example 2).

A.5 Proof of Theorem 1

Let $V_1 \geq \ldots \geq V_n > 0$ denote contestants' valuations, and let $R > 0$ be the returns-to-scale parameter of the Tullock technology. Suppose that there exists an explicit

[40] We do not know, however, if that trick works for the Tullock contest also if $n \geq 4$.

formula

$$x_i^* = f_{n,R}(V_1, \ldots, V_n)$$

for the equilibrium effort for some player $i \in \{1, \ldots, n\}$, such that $f_{n,R}$ can be computed from the primitives $V_1, \ldots, V_n$ using basic arithmetic operations and by extracting roots alone. Then, in particular for $R = \frac{1}{2}$, we have a representation

$$x_i^* = f_{n,R=\frac{1}{2}}(V_1, \ldots, V_n).$$

From the above, we know that

$$x_i^* = \frac{X^* V_i^2}{(2X^* + V_i)^2},$$

where X^* is the unique positive solution of

$$0 = \frac{V_1}{2X + V_1} + \ldots + \frac{V_n}{2X + V_n} - 1.$$

Rewriting yields

$$\frac{\left\{ \sum_{i=1}^{n} V_i \prod_{j \neq i} (2X + V_j) \right\} - \prod_{i=1}^{n} (2X + V_i)}{\prod_{i=1}^{n} (2X + V_i)} = 0. \tag{A.5}$$

For $k \in \{0, \ldots, n\}$, let $\sigma_k(V_1, \ldots, V_n)$ denote the *elementary symmetric polynomial* of degree k in the variables $V_1, \ldots, V_n$. That polynomial may be defined recursively by

$$\sigma_k(V_1, \ldots, V_n) = \sigma_k(V_1, \ldots, V_{n-1}) + V_n \sigma_{k-1}(V_1, \ldots, V_{n-1}),$$

where the initial condition is

$$\sigma_k() = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to check by induction that

$$\begin{aligned}\prod_{i=1}^{n}(2X+V_i) &= \textstyle\sum_{k=0}^{n}\sigma_k(V_1,\ldots,V_n)(2X)^{n-k},\\ \sum_{i=1}^{n}V_i\prod_{j\neq i}(2X+V_j) &= \textstyle\sum_{k=0}^{n}k\sigma_k(V_1,\ldots,V_n)(2X)^{n-k}.\end{aligned}$$

The numerator in (A.5) is, therefore, given by

$$g_n(X;V_1,\ldots,V_n)=\textstyle\sum_{k=0}^{n}(1-k)\sigma_k(V_1,\ldots,V_n)(2X)^{n-k}.$$

The following lemma is commonly used to compute Galois groups modulo prime numbers, but it can be used also in our setting.

Lemma A.2 (Specialization) *Let $\mathfrak{R}$ be an integral domain with unity allowing for unique factorization into primes and let $\mathfrak{P}$ be a prime ideal of $\mathfrak{R}$. If $g(X)$ is a polynomial with coefficients in $\mathfrak{R}$, and the image $\overline{g}(X)$ of $g(X)$ under the canonical epimorphism $\mathfrak{R}\to\overline{\mathfrak{R}}=R/\mathfrak{P}$ has no multiple roots, then the Galois group of $\overline{g}(X)$ is a subgroup of $g(X)$.*

Proof. See van der Waerden (1931, §61). $\square$

The next lemma completes the proof of Theorem 1.

Lemma A.3 *The following statements are true:*

(i) $g_n(X;V_1,\ldots,V_n)$ *does not have multiple roots; moreover,* $g_n(0;V_1,\ldots,V_n)\neq 0$*;*

(ii) $g_n(X;V_1,\ldots,V_{n-1},0)=2X\cdot g_{n-1}(X;V_1,\ldots,V_{n-1})$*;*

(iii) the Galois group of $g_{n-1}(X;V_1,\ldots,V_{n-1})$ *over* $\mathbb{Q}(V_1,\ldots,V_{n-1})$ *is a subgroup of the Galois group of* $g_n(X;V_1,\ldots,V_n)$ *over* $\mathbb{Q}(V_1,\ldots,V_n)$*;*

(iv) $g_5(X; V_1, V_2, V_3, V_4, V_5)$ *has the Galois group* S_5 *over* $\mathbb{Q}(V_1, V_2, V_3, V_4, V_5)$*;*

(v) the Galois group $g_n(X; V_1, \ldots, V_n)$ *over* $\mathbb{Q}(V_1, \ldots, V_n)$ *contains* S_5 *as a subgroup.*

Proof. (i) This was shown in the body of the paper. (ii) By the recursive definition of the elementary symmetric polynomials,

$$\sigma_k(V_1, \ldots, V_{n-1}, 0) = \begin{cases} \sigma_k(V_1, \ldots, V_{n-1}) & \text{if } k < n \\ 0 & \text{if } k = n. \end{cases}$$

Hence,

$$\begin{aligned} g_n(X; V_1, \ldots, V_{n-1}, 0) &= \textstyle\sum_{k=0}^{n}(1-k)\sigma_k(V_1, \ldots, V_{n-1}, 0)(2X)^{n-k} \\ &= \textstyle\sum_{k=0}^{n-1}(1-k)\sigma_k(V_1, \ldots, V_{n-1})(2X)^{n-k} \\ &= 2X \cdot \textstyle\sum_{k=0}^{n-1}(1-k)\sigma_k(V_1, \ldots, V_{n-1})(2X)^{n-1-k} \\ &= 2X \cdot g_{n-1}(X; V_1, \ldots, V_{n-1}), \end{aligned}$$

as has been claimed. (iii) Consider the integral domain $\mathfrak{R}_n = \mathbb{Z}[V_1, \ldots, V_n]$, i.e., the ring of polynomials in the variables $V_1, \ldots, V_n$ with rational coefficients. In $\mathfrak{R}_n$, we have the prime ideal $\mathfrak{P}_n = \langle V_n \rangle$ generated by the polynomial V_n. There is an isomorphism

$$\overline{\mathfrak{R}}_n = \mathfrak{R}_n / \mathfrak{P}_n \simeq \mathfrak{R}_{n-1}.$$

Moreover, the image of $g_n(X; V_1, \ldots, V_n)$ under the canonical epimorphism $\mathfrak{R}_n \to \overline{\mathfrak{R}}_n$ is

$$\overline{g}_n(X; V_1, \ldots, V_n) = g_n(X; V_1, \ldots, V_{n-1}, 0) = 2X \cdot g_{n-1}(X; V_1, \ldots, V_{n-1}),$$

and by part (i) free of multiple zeros. By the specialization lemma (Lemma A.2, the Galois group of $g_{n-1}(X; V_1, \ldots, V_{n-1})$ is indeed a subgroup of the Galois group of $g_n(X; V_1, \ldots, V_n)$. (iv) We have seen above that $g_5(X)$ has the Galois group S_5 over $\mathbb{Q}$. By another application of the Lemma A.2, this implies that S_5 is a subgroup of the Galois group of $g_5(X; V_1, V_2, V_3, V_4, V_5)$ over $\mathbb{Q}(V_1, V_2, V_3, V_4, V_5)$. Given that $g_5(X; V_1, V_2, V_3, V_4, V_5)$ is of degree five in X, this implies the claim. (v) The claim follows via induction from parts (iii) and (iv). $\square$

It follows from the above that the equilibrium efforts of any player cannot be expressed from the valuations vector using basic arithmetic operations and by extracting roots. The argument for the winning probability and equilibrium payoff of any individual contestant is now entirely analogous and, therefore, omitted.

A.6 The case of rational R

Suppose that $R = \frac{p}{q}$, with $0 < p < q$ integers. Then, the first-order conditions read

$$x_i^{(2p-q)/q} - x_i^{(p-q)/q} X^{p/q} + \frac{qX^{2p/q}}{pV_i} = 0,$$

where $X = (x_1^{p/q} + \ldots + x_n^{p/q})^{q/p}$. Letting $y_i = x_i^{1/q}$, we obtain

$$y_i^{2p-q} - y_i^{p-q} Y^p + \frac{qY^{2p}}{pV_i} = 0, \tag{A.6}$$

with $Y = (y_1^p + \ldots + y_n^p)^{1/p} = X^{1/q}$. Plugging the explicit expression for Y into equation (A.6) and rearranging, we arrive at the system of polynomial equations

$$pV_i(y_1^p + \ldots + y_{i-1}^p + y_{i+1}^p + \ldots + y_n^p) - qy_i^{q-p}(y_1^p + \ldots + y_n^p)^2 = 0. \tag{A.7}$$

We illustrate these equations with an example.

Example A.2 *Let $n = 2$, $p = 1$, and $q = 2$. Then, system (A.7) reads*

$$\begin{aligned} V_2 y_1 - 2y_2(y_1 + y_2)^2 &= 0, \\ V_1 y_2 - 2y_1(y_1 + y_2)^2 &= 0. \end{aligned}$$

Following Kubler et al. (2014), one may use Mathematica to compute a parameterized Gröbner basis, which yields

$$x_i^* = \frac{\sqrt{V_1 V_2}}{2(\sqrt{V_1} + \sqrt{V_2})^2} \cdot V_i,$$

consistent with Proposition 3.

A.7 Computer-assisted parts of the proofs

The computation of the Galois group in Example 3 has been accomplished with the help of the *Sage* software tool. The tool is freely accessible as a web application on sagecell.sagemath.org. The following code has been used to check that $g_1(X)$ is irreducible in the field of rational numbers $\mathbb{Q}$. The last line is the output produced by the tool.

```
R.<X>=PolynomialRing(QQ)
(8*X^5-170*X^3-450*X^2-411*X-120).is_irreducible()
True
```

Similarly, the code below allows to compute the Galois group of $g_1(X)$, where again, the last line is the *Sage* output.

```
R.<X> = PolynomialRing(QQ)
g(X) = 8*X^5-170*X^3-450*X^2-411*X-120
K.<a> = NumberField(g(X))
K.galois_group()
Galois group 5T5 (S5) with order 120
```

To replicate the computer-assisted parts of our derivations, it suffices to copy any of the above two code snippets (deleting accidental spaces at the beginning of each input line and dropping the output line entirely) to the *Sage* input box and click on the evaluate button.

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